THE ELECTROSTATIC FORCE ON A DIELECTRIC SPHERE RESTING ON A CONDUCTING SUBSTRATE

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The electrostatic force of removal is calculated for a sphere in contact with a grounded plane in an externally applied electric field that is normal to the plane. The electrostatic force is given by the sum of the Lorentz force QE_o, where Q is the free charge on the sphere and E_o is the applied electric field, and the electrical force between the sphere and the plane. The force between the sphere and the plane can be described by the interaction between the bound and free charges on the sphere, whose distribution is strongly influenced by the polarization of the sphere, and their images in the plane. The polarization charge distribution of the sphere is described by a linear multipole expansion. The multipole terms are calculated by a simple, iterative, self-consistent scheme, in which the externally applied field and the image charges induce the polarization of the sphere. The net electrostatic force on the sphere is given by the sum of the force on each linear multipole in the expansion. Two novel results of this force computation are found. The force on the higher order multipoles increases with the applied electric field more rapidly than the Lorentz force. For a given charge level, a field magnitude exists above which the net electric force is adhesional. Furthermore, an optimum charge level exists that minimizes the field required for electrostatic removal.

INTRODUCTION

The electrostatic force on a charged insulating particle resting on a plane conductor is important in a wide variety of applications including the electrostatic transfer of toner from a photoconductor in the electrophotographic cycle.[1] High transfer efficiency is achieved when the electrostatic force of removal greatly exceeds the adhesion force. The adhesional forces acting on an insulating particle resting on a substrate have been broadly classified elsewhere.[2,3] In order of decreasing strength they include chemical forces, the double layer force, the van der Waal/London dispersive force, and the gravitational force. The image force is often included as another source of adhesion, which is appropriate for charged particles with no externally applied field. In the case of electrostatic transfer or removal, the image force is a strong function of the applied electric field, as shown in this paper. Therefore, the image force should be considered as part of the net electrostatic force of removal.

Current flow determines the charge distribution[4] and the electrostatic force on particles with nonzero conductivity.[4,5] Electrostatic computations on a charged dielectric sphere with a fixed initial charge distribution and induced polarization is appropriate for insulating toner particles in the electrophotographic cycle because the charge relaxation time for toners is very long (hours or more) compared with the typical residence time on the photoconductor (seconds). Thus, the charge redistribution due to current flow within toner particles is negligible.

The image force on a point charge Q located at a distance R from an infinite plane conductor at zero potential is usually analyzed by use of the method of images and Coulomb's law. The resulting image force is given by

$$F_I = \frac{1}{4\pi\varepsilon_m} \frac{Q^2}{(2R)^2} \qquad (1)$$

Computing the force on a charged, dielectric sphere of radius R and permittivity ε_p in a medium of permittivity ε_m, resting in contact with an infinite grounded plane (Figure 1, s = 0) is greatly complicated by the induced polarization charge on the sphere. The sphere is polarized by the field produced by the image charge in the conducting plane as well as any externally applied electric field. For an isolated sphere in a uniform field, the polarization produces no net force. For a dielectric sphere near a grounded plane, however, the field due to the image charges is manifestly nonuniform. In that case the simple expression for the image force on the sphere must be modified to include the dielectrophoretic (DEP) force.

The dielectrophoretic effect has been modelled by representing the dielectric sphere by a simple dipole.[6] In the uniform field approximation,[6,7] the dipole moment is given by

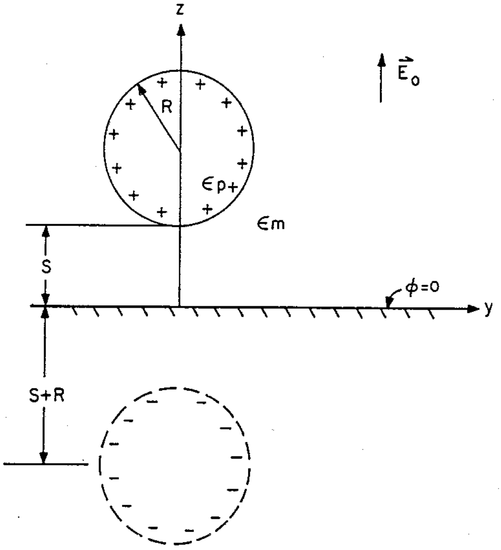

Figure 1. Dielectric sphere, of permittivity ε_p and radius R, a distance s from the conducting plane, located a z = 0, in a medium of permittivity ε_m. Also shown is an applied field E_o, normal to the plane.

$$\overline{p}_{eff} = 4\pi\varepsilon_m \frac{\varepsilon_p - \varepsilon_m}{\varepsilon_p + 2\varepsilon_m} R^3 \overline{E} \quad (2)$$

where the value of $\overline{E}$ at the location of the center of the sphere, with the sphere removed, is used. The nonuniformity of the electric field is then used for the force expression

$$\overline{F}_{DEP} = (\overline{p}_{eff} \cdot \nabla)\, \overline{E} \quad (3)$$

In the case of a sphere that is resting in contact with a conducting surface this approach is clearly inadequate. Because the image charge is located at z = -R, only 2R from the center of the particle, the variation of the field is large compared with the dimension of the particle. It has been shown that the simple approach to analyzing DEP can be corrected by the use of higher order multipoles.[8,9] In this case, the charge distribution for the sphere can be represented by a linear multipole expansion, located for convenience at the center of the sphere.

The solution set of multipoles and their images is found by simultaneously solving a set of three electrostatic equations. The first equation relates the source multipole, which describe the charge distribution of the dielectric sphere, to the generating field, which is comprised of the field due to the image multipoles and the externally applied field. The second equation describes the field in the half-space outside of the grounded plane due to the image multipoles by a multipole expansion of the potential. The third equation relates the image multipoles located at z = -R to the source multipoles located at z = +R by the method of images.

THEORY

Multipole Calculation

The potential outside of any localized charge distribution can be written as an expansion in spherical harmonics, located at the origin of the coordinate system,

$$\phi(\overline{r}) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} B_{\ell,m} \frac{Y_{\ell,m}(\theta,\phi)}{r^{\ell+1}} \quad (4)$$

where the coefficients $B_{\ell,m}$ contain the multipole moments.[10] In this paper only the axially symmetric case is considered because the applied field is normal to the conducting plane (Figure 1). In the case of axial (azimuthal) symmetry, the spherical harmonics can be replaced by Legendre polynomials $P_n(\cos\theta)$

$$\phi(r,\theta) = \sum_{n=0}^{\infty} \frac{p^{(n)} P_n(\cos\theta)}{4\pi\varepsilon_m r^{n+1}} \quad (5)$$

and the $p^{(n)}$ are defined to be linear multipoles.[11] Linear multipoles can be generated by combinations of point charges as shown in Figure 2. A dipole is generated by inverting the charge of the monopole and displacing it by a distance d from the positive monopole. A linear quadrupole is generated by the same operation -- inverting the charge of a dipole and displacing it from a positive dipole. An octupole is generated from two quadrupoles and so forth. Note that in each case the pole whose charge

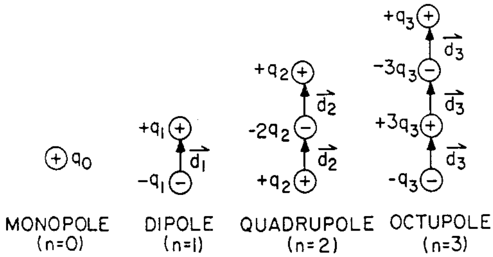

Figure 2. Linear multipoles, constructed from point charges.

was reversed is displaced in the -z direction so that positive linear multipoles have the characteristic that the point charge furthest in the +z direction is positive. In this point charge construction, the magnitude of the q_n and the d_n determines the magnitude of the n^{th} multipole. With the appropriate linear multipoles in Equation 5, the potential in space due to the dielectric sphere can be specified. A second expansion, closely related to the first, is used for the image charges.

The multipole moments induced in a dielectric sphere by an externally applied electric field have been calculated by Jones[8,9] in terms of the axial field E and its derivatives, evaluated at the location of the center of the sphere with the sphere removed.

$$p_i^{(n)} = \frac{4\pi\varepsilon_m(\varepsilon_p - \varepsilon_m)\, R^{2n+1}}{(n-1)!\, [n\, \varepsilon_p + (n+1)\varepsilon_m]} \frac{\delta^{n-1} E_z}{\delta z^{n-1}}, \quad n = 1,2,3,... \tag{6}$$

The monopole moment $p^{(0)} = Q$ is given by the net charge on the sphere. If we use the definition

$$\frac{\delta^0 E_z}{\delta z^0} = E_z \tag{7}$$

then Equation 2 is simply a special case of Equation 6.

Consider an initially unpolarized, dielectric sphere with charge Q whose center is located at z = +R, as shown in Figure 1 with s=0. Associated with the ground plane is an image charge located at z = -R. The axially symmetric field evaluated along the z-axis, with the sphere removed, is given by

$$E_z = \frac{1}{4\pi\varepsilon_m} \frac{-Q}{(z+R)^2} \tag{8}$$

Inserting Equation 8 into Equation 6 shows that the image monopole will induce an infinite set of linear multipoles $p^{(n)}$ at z = +R in the dielectric sphere. This will, in turn, require an infinite set of image multipoles $p_i^{(n)}$ located at z = -R to preserve the boundary condition along z = 0. Thus, the potential due to the charge distribution on the grounded plane is given, according to Equation 5, by

$$\phi(r,\theta) = \sum_{n=0}^{\infty} \frac{p_i^{(n)} P_n(\cos\theta)}{4\pi\varepsilon_m\, r^{n+1}} \tag{9}$$

The axially symmetric field along the z axis is given by $E = -\nabla\phi$ evaluated at $\theta = 0$, $r = z + R$.

$$E_z = \sum_{n=0}^{\infty} \frac{(n+1)\, p_i^{(n)}}{4\pi\varepsilon_m (z+R)^{n+2}} + E_o \qquad (10)$$

Also included in Equation 10 is any externally applied field E_o that may be present.

The Method of Images

The image multipoles must be related to the source multipoles according to the method of images. The point charge representation illustrated in Figure 2 makes the construction of image multipoles straightfoward as shown in Figure 3. The method of images actually involves two operations, a spatial inversion about the z = 0 plane and charge inversion. Thus for a point source charge q located at z, the image charge is given by -q located at -z. The effect of these two operations on linear multipoles depends on the symmetry of the multipole with respect to the transformations. The charge inversion operator $\mathcal{C}$ produces an opposite polarity multipole.

$$\mathcal{C}(p^{(n)}) = -p^{(n)} \qquad (11)$$

The effect of the spatial inversion operator $\mathcal{M}$, however, depends upon the mirror symmetry of the linear multipole.

$$\mathcal{M}(p^{(n)}) = (-1)^n p^{(n)} \qquad (12)$$

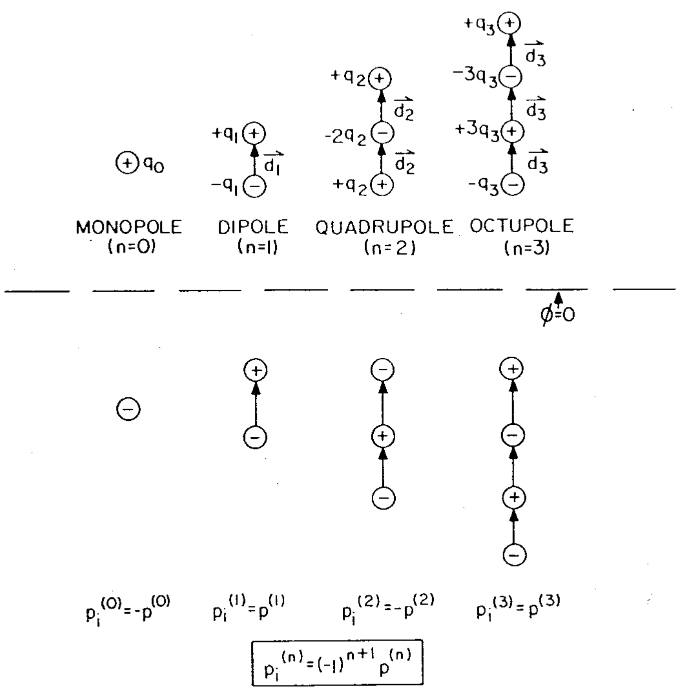

Figure 3. Linear multipoles and their images.

Combining the operators gives the complete method of images transformation

$$i = c * m \qquad (13)$$

$$p_i^{(n)} = i(p^{(n)}) = (-1)^{n+1} p^{(n)} \qquad (14)$$

Force Calculation

The electrical force on the dielectric sphere is given[9] by

$$F_e = \sum_{n=0}^{\infty} \frac{p^{(n)}}{n!} \frac{\delta^n E_z}{\delta z^n} \qquad (15)$$

Inserting Equations 10 and 14 into Equation 15, and doing the differentiation term by term,

$$F_e = p^{(0)} E_o + \frac{1}{4\pi\varepsilon_m} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^{n+k+1} (n+k+1)!\, p^{(n)} p^{(k)}}{n!\; k!\; (z+R)^{n+k+2}} \qquad (16)$$

By solving Equations 6, 10, and 14 simultaneously for z = R, the solution set of multipoles necessary to evaluate Equation 16 are determined for the case of a dielectric sphere of radius R touching a conducting substrate. Equation 16 is evaluated using this set of multipoles and z = R. Note that the Lorentz force QE_o appears as the leading term of Equation 16. Similarly, the image force, Equation 1, appears as the n = k = 0 term.

RESULTS

The details for solving Equations 6, 10, and 14 are given elsewhere.[12] The solution is restricted to a finite number of multipoles, $N \leq 64$. Once the solution set $p^{(0)} \ldots p^{(N)}$ is found, Equation 16 is evaluated to find the net electrostatic force on the sphere.

Before analyzing the case of electrostatic removal, consider a charged, dielectric sphere a distance s from a conducting plane with no applied field, in a vacuum. Figure 4 is a logarithmic plot of the force attracting the sphere towards the plane as a function of the separation distance calculated using Equation 16. Note that for a sphere of relative permittivity $k_p = 1$ ($k_p = \varepsilon_p / \varepsilon_0$) no polarization is possible, this case represents Equation 1. At large separations, polarization effects are negligible and the force of attraction for all k_p is given by Equation 1. As the separation becomes small, the field produced by the image charge is sufficient to polarize the sphere for $k_p > 1$ and significantly alter the image force. Davis[13] analyzed this problem by numerically solving Laplace's equation in bispherical coordinates. His results are also given in Figure 4 for comparison.

A force of attraction between an uncharged sphere and a ground plane is induced by a strong external field because it polarizes the sphere. This case is shown in Figure 5 along with Davis's results for the identical problem. At large separations, only the dipole interactions are significant. As the separation becomes small, force contributions from higher order multipoles become important.

The problem of the electrostatic removal force of a dielectric sphere in contact with a conducting plane combines the features of Figures 4 and 5. That is, the attractive electrical force increases with the charge Q

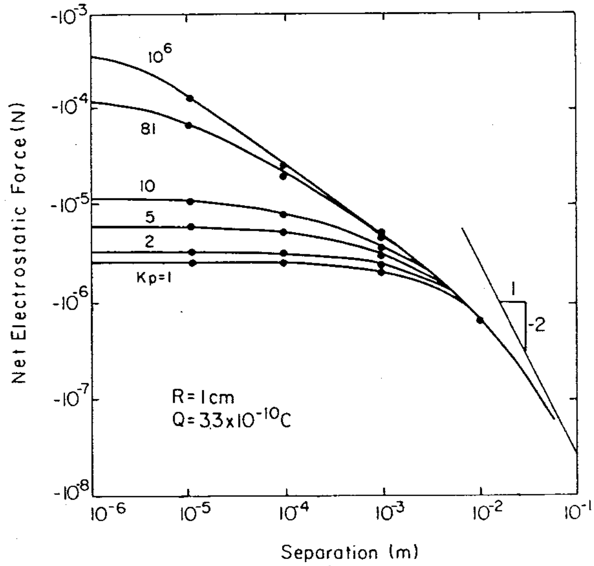

Figure 4. The net electrostatic force on a dielectric sphere, of charge $Q = 3.3 \times 10^{-10}$ C, in a medium of permittivity ε_0 vs. the separation between the sphere and a ground plane. Shown are the results from the multipole calculation for several relative permittivities of the sphere (solid lines). Also shown are the results from a solution to Laplace's equation in bispherical coordinates from Reference 13 (circles).

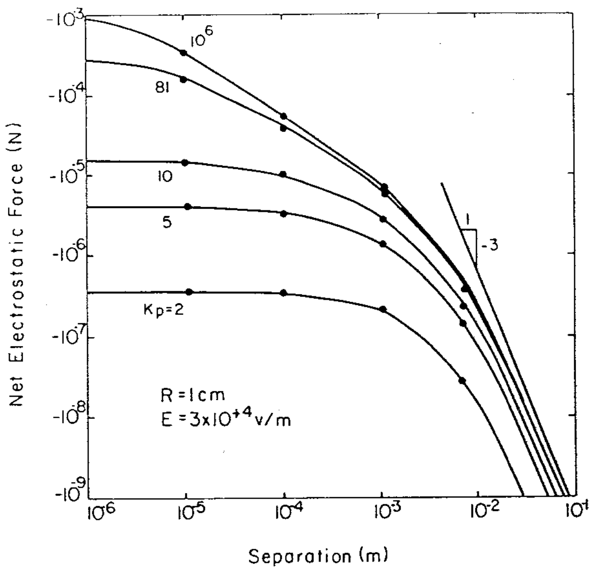

Figure 5. The net electrostatic force on an uncharged dielectric sphere in a medium of permittivity ε_0 with an applied field $E_0 = 3 \times 10^4$ V/m vs. the separation between the sphere and a ground plane. Shown are the results from the multipole calculation for several relative permittivities of the sphere (solid lines). Also shown are the results from a solution to Laplace's equation in bispherical coordinates from Reference 13 (circles).

(Equation 1) and with the dielectric constant ε_p (Figure 4) as well as with the applied field E_0 (Figure 5). On the other hand, the Lorentz force pulling the sphere away from the plane goes as QE_0 and is independent of ε_p. Thus, we can expect to find that the force of removal is a complicated function of Q, E_0, and ε_p.

The net electrical force may be plotted as a function of the particle charge and the applied electric field using equi-force contour plots as shown in Figure 6. Identified are the four regimes of importance.

Polarization Adhesion: $F_e < 0$
Image Adhesion: $F_e < 0$
Mechanical Removal: $0 < F_e < F_A$
Electrostatic Removal: $F_A < F_e$

where F_e is the net electrical force, Equation 16, and F_A is the adhesion force given by

$$F_A = F_V + F_G \qquad (17)$$

where F_V is the van der Waals force between a sphere and a plane.

$$F_V = \frac{h\,\omega_0}{8\pi z_0^2} R \qquad (18)$$

where $h\omega_0$ is the Lifshitz-van der Waals constant and z_0 is the distance of closest approach between the sphere and plane. The gravitational force is given by

$$F_G = \frac{4}{3}\pi R^3 \rho \qquad (19)$$

where ρ is the volume density of the sphere.

In the Polarization Adhesion regime, the applied electric field is large, but the particle charge is small. The net electrical force is dominated by the attraction of the induced polarization charge (multipoles)

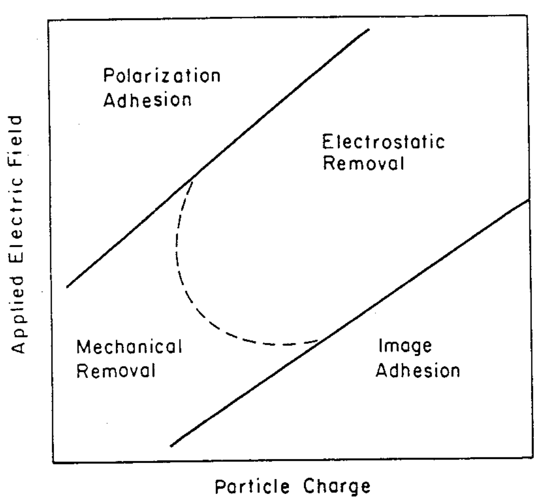

Figure 6. Sketch of the equi-force contours for the combinations of particle charge and applied electric field identifying the four regimes of interest.

to their images in the grounded plane. Thus, $F_e < 0$, or towards the ground plane. The existence of the Polarization Adhesion regime is the principal new result of this work. Since the magnitude of multipole interactions decreases rapidly with separation distance, removal by mechanical vibration or other means is possible if sufficient separation is achieved. Nevertheless, removal by mechanical means may not be very effective in the Polarization Adhesion regime because separation distances on the order of a particle radius must be achieved.

In the Image Adhesion regime, the applied electric field is small and the particle charge is high. The Coulombic force between the monopole ($p^{(0)} = Q$) and its image in the grounded plane dominates.

In the Mechanical Removal regime, both the applied field and the particle charge are small. The electrostatic force favors removal but is smaller than the force of adhesion (van der Waals and gravitational). Mechanical vibrations or other means of separating the particle from the plane momentarily are required for removal to occur.

In the Electrostatic Removal regime, both the particle charge and applied field are large. The Lorentz force dominates and a particle is removed electrostatically from the grounded plane. This condition is achieved in the electrophotographic cycle to accomplish high toner transfer efficiency. Of course, the location of the demarcation line between the Mechanical Removal regime and the Electrostatic Removal regime will depend upon the magnitude of F_A.

Computed results corresponding to Figure 6 are given in Figures 7 - 9 for a 10 μm radius sphere and a range of relative permattivities, as labelled on the figures. In each case the medium is vacuum, $\varepsilon_m = \varepsilon_0$ and the sphere is taken to be resting on the conducting substrate. The equi-force contours are given by the force ratio

$$\mathscr{f} = \frac{F_e}{(F_A + F_G)} \qquad (20)$$

The van der Waals force was calculated for Equation 18 using[3,14] $\hbar\omega_0 = 3$ eV and $z_0 = 0.6$ nm, the gravitational force was calculated from Equation 19 using $\rho = 2.5 \times 10^3$ kg/m^3 but is negligible for this size sphere.

The values for E_0 range from the upper limit of field induced emission at approximately $E_0 = 10^8$ V/m to $E_0 = 10^5$ V/m. Naturally, fields above 3×10^7 V/m or higher are possible for 10 μm spheres in air only if air gaps are kept small according to the Paschen curve[15].

Several observations may be made from Figures 7 - 9:

1. The monopole's attraction to its image in the ground plane (the familiar point charge image force, Equation 1) dominates the Lorentz force in the Image Adhesion regime. The transition into removal regimes is not strongly dependent on the relative permittivity of the sphere. As seen in Figure 4, the electrical force varies by less than an order of magnitude over a $2 < \varepsilon_p < 10$. The $\mathscr{f} = 0$ contour bordering the Image Adhesion regime has a slope of unity, and to first order is independent of the particle permittivity. This can be demonstrated analytically if we consider only the $n = k = 0$ term in Equation 16. Thus, along the $\mathscr{f} = 0$ line

$$\mathscr{f} = \frac{QE_0 - \dfrac{1}{4\pi\varepsilon_m}\dfrac{Q^2}{(2R)^2}}{F_A} = 0 \qquad (21)$$

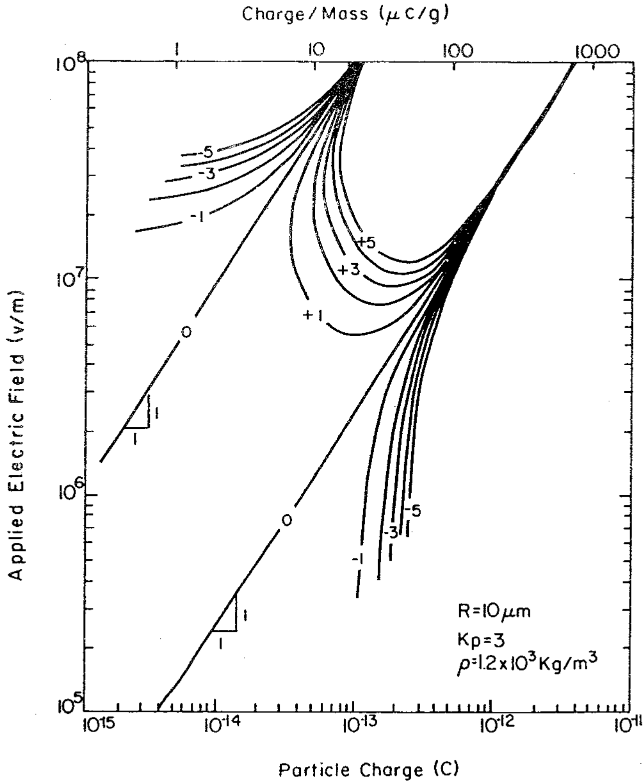

Figure 7. Equi-force contours for a 10 μm radius particle of relative permittivity $k_p = 3$, in units of adhesion force $F_A = 5 \times 10^{-7}$ N. The zero force lines indicate the boundary between the two adhesion regimes and the two removal regimes. The +1 force line is the boundary between the Mechanical Removal regime and the Electrostatic Removal regime. Also shown are charge/mass values calculated assuming a particle mass density of 1.2×10^3 kg/m, which is appropriate for Kodak Ektaprint™ toner.

therefore,

$$E_0 = \frac{1}{4\pi\varepsilon_m} \frac{Q}{(2R)^2} \qquad (22)$$

2. The attraction of the induced polarization charges to their images in the ground plane dominates both the Lorentz force and other adhesion forces in the Polarization Adhesion regime. The transition between Polarization Adhesion and the removal regimes is very sensitive to the relative permittivity of the sphere. As seen in Figure 5, the electrical force at close spacings varies by over an order or magnitude for $2 < \varepsilon_p < 10$. The $\mathcal{F} = 0$ contour bordering the Polarization Adhesion regime has a slope of unity indicating[16] that the terms from Equation 16, along that contour, can be grouped into two factors, one proportional to QE_0 and the other proportional to Q^2, as in Equation 21 leading to a relationship similar to Equation 22.

3. The transition from the Image Adhesion regime to the Electrostatic Removal regime is very abrupt for a highly charged sphere as compared to a lower charged sphere's transition from Mechanical Removal to Electrostatic Removal.

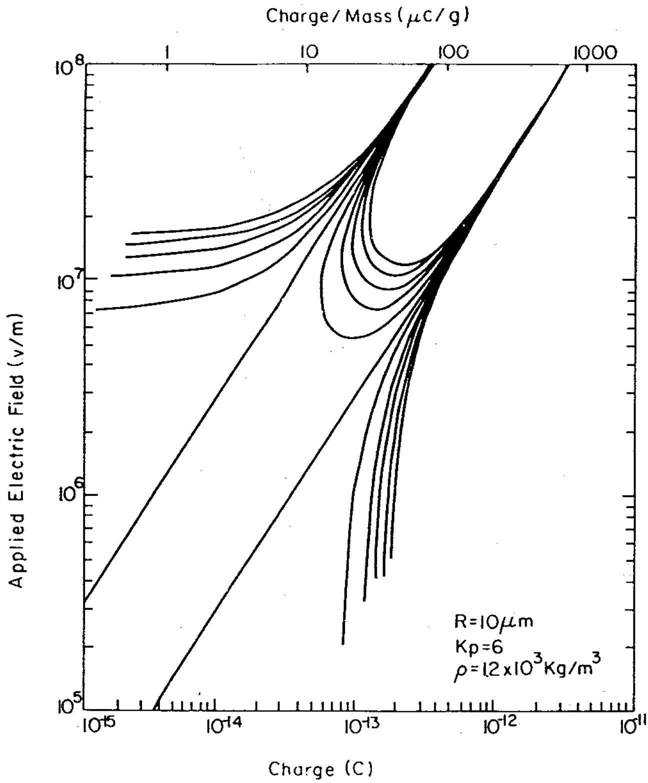

Figure 8. Equi-force contours for a 10 μm radius particle of relative permittivity $k_p = 6$, in units of adhesion force $F_A = 5 \times 10^{-7}$ N. The zero force lines indicate the boundary between the two adhesion regimes and the two removal regimes. The +1 force line (cf. Fig. 7) is the boundary between the Mechanical Removal regime and the Electrostatic Removal regime. Also shown are charge/mass values calculated assuming a particle mass density of $1.2 \times 10^3 kg/m^3$, which is appropriate for Kodak Ektaprint™ toner.

4. An optimum particle charge exists that minimizes the applied electric field required for electrostatic removal, corresponding to the minimum of the $\mathscr{f}$ = +1 contour. The optimum charge is insensitive to the relative permittivity of the sphere. For a 10 μm radius particle, the optimum charge is $\sim 1 \times 10^{-13}$ C resulting in an optimum charge to mass ratio of ~16 μC/g (mass density = 1.2×10^3 kg/m^3). In the vicinity of the optimum charge and minimum electric field, small changes in charge or applied field result in relatively small changes in force. The Lorentz force and the net electric polarization vary together. Of course, the optimum charge and field depend strongly upon the magnitude of F_A.

CONCLUSIONS

A new, heuristic method of calculating the net electrostatic force on a charged, dielectric sphere resting on a grounded plane in an externally applied electric field is presented. The polarization charge density, which can have a dramatic effect on the net force, is represented by a series of linear multipoles.

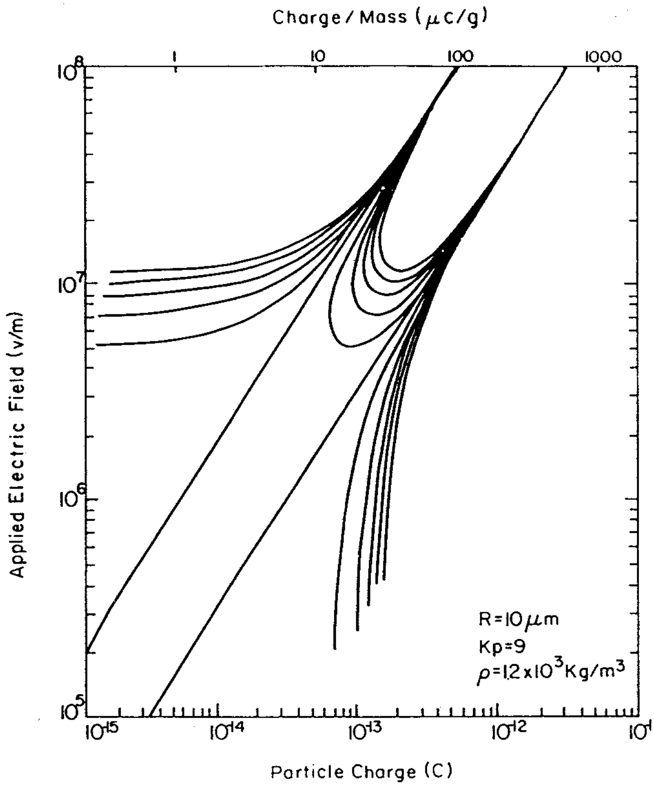

Figure 9. Equi-force contours for a 10 μm radius particle of relative permittivity $k_p = 9$, in units of adhesion force $F_A = 5 \times 10^{-7}$ N. The zero force lines indicate the boundary between the two adhesion regimes and the two removal regimes. The +1 force line (cf. Fig. 7) is the boundary between th Mechanical Removal regime and the Electrostatic Removal regime. Also shown are charge/mass values calculated assuming a particle mass density of 1.2×10^3 kg/m^3, which is appropriate for Kodak Ektaprint™ toner.

Force computations made using the multipole expansion method[9] are in agreement with the results of a formal solution to Laplace's equation using eigenfunction expansions in bispherical coordinates.[13]

Using the multipole expansion method, the electrostatic removal of dielectric spheres is examined in detail. The two important results are:

1. A regime of Polarization Adhesion is identified in which the attraction between induced polarization charges and their images in the ground plane dominate. Mechanical vibration or other means of removing particles is inefficient in this regime because separation distances on the order of a particle radius must be achieved to overcome the electrostatic adhesion.

2. An optimum particle charge exists that minimizes the electric field required for removal. For a 10 μm radius particle, assuming $F_A = 5 \times 10^{-7}$ N, typical[14] of the adhesion of Kodak Ektaprint™ toner on an illuminated Ektaprint™ film loop, the optimum charge to mass ratio is 16 μC/g.

REFERENCES

1. R. M. Schaffert, "Electrophotography", p. 52, Focal Press, London, 1980.
2. H. Krupp, Adv. Colloid Interface Sci., 1, 111 (1967).
3. N. S. Goel and P. R. Spencer, Polym. Sci. Technol., 9B (Adhes. Sci. Technol.), pp. 763-829, L. H. Lee, Editor, Plenum Press, New York, 1975.
4. K. J. McLean, J. Air Pollution Control Assoc., 27, 1100 (1977).
5. P. W. Dietz and J. R. Melcher, in "Control and Dispersion of Air Pollutants: Emphasis on NO_x and Particulate Emissions", R. L. Dyers, D. W. Cooper, and W. Licht, Editors, p. 166, American Institute of Chemical Engineers, New York, 1978.
6. T. B. Jones and G. A. Kallio, J. Electrostatics, 6, 207 (1979).
7. T. B. Jones, J. Electrostatics, 6, 69 (1979).
8. T. B. Jones, in "Proc. IEEE-IAS 1984 Annual Meeting", p. 1136, IEEE, 1984.
9. T. B. Jones, J. Electrostatics, 18, 55 (1986).
10. J. D. Jackson, "Classical Electrodynamics", p. 136, John Wiley, New York, 1975.
11. J. A. Stratton, "Electromagnetic Theory", p. 172, McGraw Hill, New York, 1941.
12. K. S. Robinson and W. Y. Fowlkes, to be published, J. Electrostatics.
13. M. H. Davis, Am. J. Phys., 37, 26 (1969).
14. D. S. Rimai, (1986), unpublished data.
15. J. D. Cobine, "Gaseous Conductors", p. 164, Dover, New York, 1958.
16. L. Marks, (1987), personal communication.