Basis Models for Complex Electrostatic Simulations

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I. INTRODUCTION

Imagine that we have a collection of conducting objects in a region under electrostatic equilibrium. We impose a Dirichlet boundary condition by assigning to each object a potential voltage. We solve for the charge distribution that satisfies the boundary condition by using variant of the Poisson equation. There are only a few cases where we can do this by hand; a variety of numerical techniques are used for everything else.

We use the Method of Moments as described by Harrington to approximate the charge distribution [1]. We discretize the surfaces of the objects into a finite collection of geometric elements. Each geometric element has a known potential voltage (determined by the boundary condition) and an unknown charge. We solve for the unknown charges by defining and solving a system of linear equations. In other words, we want to solve the matrix equation, $\mathbf{A}\boldsymbol{\rho} = \mathbf{V}$, where $\mathbf{A}$ is the coefficient matrix, $\boldsymbol{\rho}$ is the unknown charge distribution, and $\mathbf{V}$ is the specified boundary condition. Note that the coefficient matrix is dependent only on the geometry of the model. We can compute $\boldsymbol{\rho}$ as $\boldsymbol{\rho} = \mathbf{A}^{-1}\mathbf{V}$.

We are free to let the objects move and the boundary condition change. We solve for the charge distribution at a series of discrete time steps [2]. In general, we can no longer assume electrostatic equilibrium because the moving charge distribution produces magnetic fields. However, in practice, if the objects are moving slowly enough and the boundary condition is changing slowly enough, the intensity of the induced magnetic fields is small enough that we can safely ignore them. Under this simplifying assumption, we create and compute a time series of quasistatic frames. We put these frames together to produce a dynamic electric field model.

As the complexity of models increases, so does the time needed to compute them. In fact, the time required to compute a single frame of a dynamic model is proportional to the cube of the number of elements in the frame. This paper introduces a technique based on the Principle of Superposition that reduces the amount of time needed to compute large dynamic models with many frames.

II. MATHEMATICAL FRAMEWORK

Imagine that we have a collection of models that have the same geometry, and therefore the same coefficient matrix, $\mathbf{A}$, but different boundary conditions, $\mathbf{V_1}, \mathbf{V_2}, \ldots, \mathbf{V_n}$, and we have solved for their corresponding charge distributions, $\boldsymbol{\rho}_1, \boldsymbol{\rho}_2, \ldots, \boldsymbol{\rho}_n$. We assume that $\mathbf{V_1}, \mathbf{V_2}, \ldots, \mathbf{V_n}$ are linearly independent. Now consider another model that has the same geometry, and therefore the same coefficient matrix, $\mathbf{A}$, as the other models, but a different boundary condition, $\mathbf{V}$, and we want to solve for its charge distribution, $\boldsymbol{\rho}$.

For real numbers, $a_1, a_2, \ldots, a_n$, if we can write $\mathbf{V}$ as $\mathbf{V} = a_1\mathbf{V_1} + a_2\mathbf{V_2} + \ldots + a_n\mathbf{V_n}$, then $\boldsymbol{\rho} = a_1\boldsymbol{\rho}_1 + a_2\boldsymbol{\rho}_2 + \ldots + a_n\boldsymbol{\rho}_n$. The proof of this is below.

$$\boldsymbol{\rho} = \mathbf{A}^{-1}\mathbf{V}$$
$$\boldsymbol{\rho} = \mathbf{A}^{-1}(a_1\mathbf{V_1} + a_2\mathbf{V_2} + \ldots + a_n\mathbf{V_n})$$
$$\boldsymbol{\rho} = a_1\mathbf{A}^{-1}\mathbf{V_1} + a_2\mathbf{A}^{-1}\mathbf{V_2} + \ldots + a_n\mathbf{A}^{-1}\mathbf{V_n}$$
$$\boldsymbol{\rho} = a_1\boldsymbol{\rho}_1 + a_2\boldsymbol{\rho}_2 + \ldots + a_n\boldsymbol{\rho}_n$$

We take advantage of this fact by creating and computing a collection of basis models. Note that all of the basis models must have the same geometry. The span of the basis models yields a model space, and every model in the model space is implicitly computed once the basis models are computed.

Now consider a model with a finite number of conducting objects. Suppose that we want there to be a specific total charge on one or several of the objects. We can use the already established facts about basis models to do this as well. Let $V_1, V_2, \ldots, V_N$ be the potential voltage on each of the objects and $Q_1, Q_2, \ldots, Q_N$ be the total charge on each of the objects. In addition, we create and compute a basis model for each object in the model. We compute the total charge on each of the objects in each of the basis models to be $Q_{i,j}$, where i is the object and j is the basis model. Let $(a_1, a_2, \ldots, a_n)$ be the coordinate of the original model in the model space. The following system of linear equations holds.

$$a_1Q_{1,1} + a_2Q_{1,2} + \ldots + a_nQ_{1,n} = Q_1$$
$$a_1Q_{2,1} + a_2Q_{2,2} + \ldots + a_nQ_{2,n} = Q_2$$
$$\ldots$$
$$a_1Q_{n,1} + a_2Q_{n,2} + \ldots + a_nQ_{n,n} = Q_n$$

This allows us to specify a combination of potential voltages and total charges as the boundary condition. We can use the system of linear equations to transform the mixed boundary condition into one in terms of just potential voltages. We can then use the basis models to compute the charge distribution.

The use of basis models provides significant savings in computation time when working with large dynamic models. They provide other benefits as well. We can study the precise relationship between the source voltages and their resulting electric fields. In addition, we can create extremely high resolution dynamic models (to the point of deriving an analytic expression for the electric field at any point in time).

III. FIRST CASE STUDY: POWER LINES

We will use a detailed line segment model of a typical section of distribution power lines. We use line segments as the geometric elements in our models because they are good at

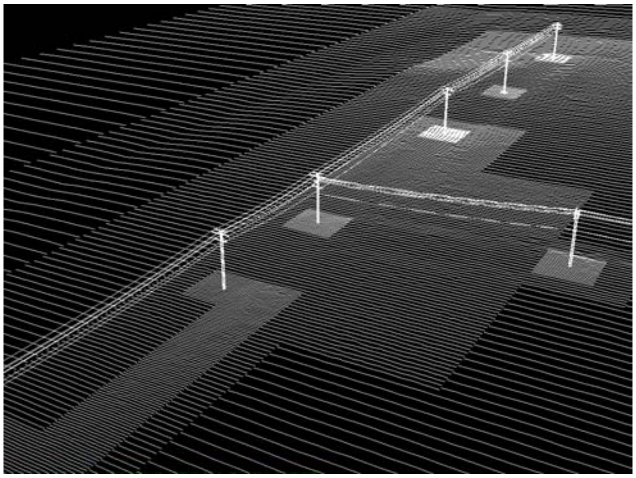

Fig. 1. Line segment model of power lines looking north

modeling rectangular objects, wires, and flat (or nearly flat) surfaces. In addition, we treat everything in our model as perfect electrical conductors. We can do this because both the voltage gradients on the wire conductors due to power currents and on the ground and power poles due to induced currents are small enough compared to the boundary condition that we can safely ignore them. The model is composed of three distinct components: the ground plane, the power poles, and the power lines (see Fig. 1). The line segments that make up the ground plane are more densely packed near the power poles and under the power lines because the electric field changes more rapidly in these areas. The power poles are modeled as a hexagonal rods and the cross bars are modeled as three dimensional rectangles. This allows us to see the effects of the electric field on different sides of the power poles. The power lines are modeled as a sequence of line segments laid head to tail. The line segments that make up the power lines are more densely packed near the cross bars to reduce any modeling artifacts. All dimensions of the model are accurate within a few centimeters.

A. Basis Models

There are two distinct sets of power lines in the model. We will operate only the west set of power lines and turn off the east set of power lines. The west set of power lines is composed of three individual wires, which we will identify as wires A, B, and C. We will construct a basis model for each one. That is, for each wire, we will create a model and ground everything except that wire. We then assign to that wire a potential voltage of 1 V. We will identify them as basis models A, B, and C.

B. Point of Computation

We want to compute the electric field at an arbitrary point along the ground as the voltages on the power lines oscillate over time. In this example, the point of computation is on the ground twenty meters south east of the center power pole. We need to compute the electric field at the point of computation in each of the three basis models. We do this by interrogating the model for the surface charge density, *D*, of the element

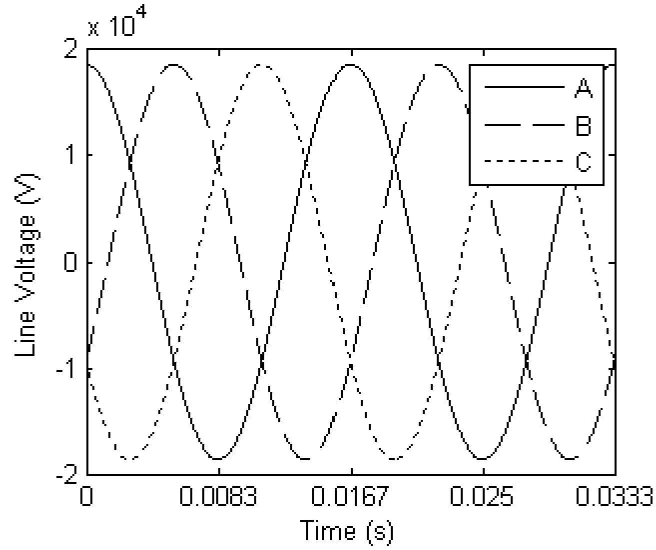

Fig. 2. Line voltages

that exists at the point of computation. We compute E as $E = D/\varepsilon$.

$E_{A,BASIS}$ = –2.6698 mV/m
$E_{B,BASIS}$ = –2.3406 mV/m
$E_{C,BASIS}$ = –2.7577 mV/m

We can use these to compute the electric field at the point of computation for any combination of the line voltages. As an example, if the line voltages of lines A, B, and C are 18668 V, –9334 V, and –9934 V, respectively, then $E_{TOTAL} = 18668E_{A,BASIS} - 9334E_{B,BASIS} - 9334V_CE_{C,BASIS} = -2.2521$ V/m.

C. The Experiment

We will choose the line voltages to mimic those on a set of ideal, three-phase, 13.2-kV, 60-Hz power lines. The line voltages are given by the following expressions (see Fig. 2).

$V_A(t)$ = (18668 V)cos(2π(60 Hz)t)
$V_B(t)$ = (18668 V)cos(2π(60 Hz)t – 2π/3)
$V_C(t)$ = (18668 V)cos(2π(60 Hz)t – 4π/3)

We need to compute the coordinate, (a(t), b(t), c(t)), of the model in the model space as the line voltages oscillate. This is trivial because we selected the potential voltages in the basis models to be 1 V.

$a(t) = V_A(t)$ / (1 V)
$b(t) = V_B(t)$ / (1 V)
$c(t) = V_C(t)$ / (1 V)

We compute the electric field at the point of computation due to each line. We then add these together to compute the total electric field at the point of computation (see Fig. 3).

$E_A(t) = a(t)E_{A,BASIS}$
$E_A(t)$ = (-49.8395 V/m)cos(2π(60 Hz)t)

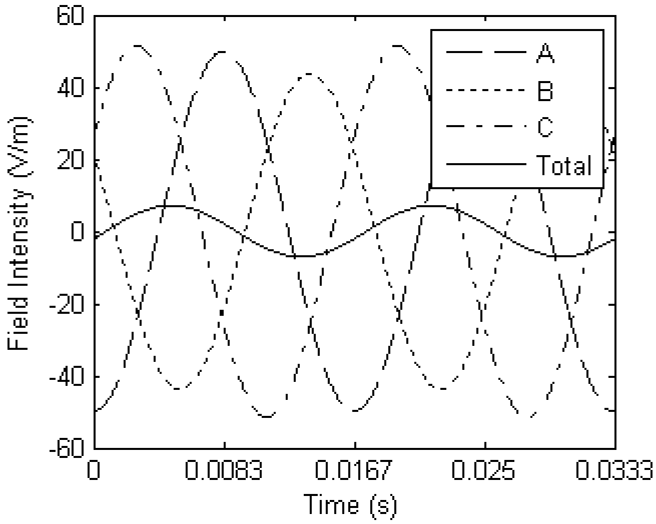

Fig. 3. Resulting electric fields

$E_B(t) = b(t)E_{B,BASIS}$
$E_B(t) = (-43.6938\text{ V/m})\cos(2\pi(60\text{ Hz})t - 2\pi/3)$

$E_C(t) = c(t)E_{C,BASIS}$
$E_C(t) = (-51.4812\text{ V/m})\cos(2\pi(60\text{ Hz})t - 4\pi/3)$

$E_{TOT}(t) = E_A(t) + E_B(t) + E_C(t)$
$E_{TOT}(t) = (7.1101\text{ V/m})\cos(2\pi(60\text{ Hz})t - 1.8931)$

Notice that the total electric field has a significantly lower amplitude and different phase angles than the individual component electric fields.

IV. SECOND CASE STUDY: HUMAN WALKING

We want to study the behavior of the electric field around a human as he walks along a straight path. We will use a dynamic model made of a time series of frames that shows the human walking. Each frame is composed of three parts: the ground plane, the sky plane, and the human. The sky plane allows us to model a vertical ambient field, such as the fair weather field of Earth. The ground and the sky are made from a dense grid of line segments. The length and width of the ground and the sky are the same and are large compared the distance between them. This is to reduce any fringing effects near the center of the ground where the human is walking. The human is modeled as a wireframe conductor and its position changes from frame to frame as the model walks. We want to look at three cases: (a) the human is grounded in a vertical ambient electric field; (b) the human is charged in no vertical ambient electric field; and (c) the human is charged in a vertical ambient electric field.

A. Basis Models

We will construct two basis models for each frame in the dynamic model. In the first basis model, we will assign to the ground and to the human a potential voltage of 0 V and to the sky a potential voltage of 1 V. In the second basis model, we will assign to the ground and to the sky a potential voltage of 0 V and to the human a potential voltage of 1 V. We then compute the solutions to both of these boundary value problems. We will identify the solutions as basis models A and B. After computing them, we need to interrogate them for a few pieces of information. Let Q_A be the total charge on the human in basis model A and Q_B be the total charge on the human in basis model B.

B. Producing a Vertical Ambient Electric Field

To produce a vertical ambient electric field, we assign the sky a potential voltage, V_{SKY}. We can compute V_{SKY} as $V_{SKY} = Ed$, where E is the desired electric field intensity and d is the distance between the sky and the ground. Note that $E \leq V_{SKY}/d$ due to fringing effects. We can minimize this error by making sure the distance between the ground and the sky is small compared to the area of both. Note that $V_{SKY} = 0$ V for zero vertical ambient electric field. Also note that V_{SKY} is independent of the frame.

C. Specifying the Total Charge on the Human

We want to specify an exact amount of charge on the human. We have stumbled upon a mixed boundary condition. We know V_{SKY} and Q_{HUMAN}, but in order to use the basis models, we need to transform the mixed boundary condition into one in terms of just potential voltages. In other words, we need to find the potential voltage of the human, V_{HUMAN}. We can compute V_{HUMAN} as $V_{HUMAN} = (Q_{HUMAN} - V_{SKY}Q_A)/Q_B$. Note that this is specific to each frame and the potential voltage of the human might change from frame to frame.

D. Putting It All Together

We can put all of this together to investigate the behavior of the electric field in each of the three cases. In the first case, we want to study the behavior when the human is grounded in

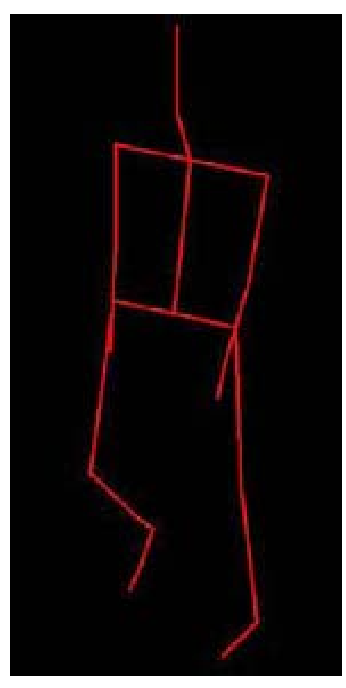
Fig. 4. Line segment model of human

a vertical ambient electric field. We choose $V_{HUMAN} = 0$ V and V_{SKY} as just discussed. In the second case, we want to study the behavior when the human is charged in no vertical ambient electric field. We choose $V_{HUMAN} = Q_{HUMAN}/Q_B$ and $V_{SKY} = 0$ V. In the third case, we want to study the behavior when the human is charged in a vertical ambient electric field. We choose $V_{HUMAN} = (Q_{HUMAN} - V_{SKY}Q_A)/Q_B$ and V_{SKY} as discussed. After choosing V_{HUMAN} and V_{SKY}, we can use the basis models to compute the charge distribution, and then using the Principle of Superposition, we can compute the electric field.

V. CONCLUSION

We have created two extremely high quality dynamic models. In the first case study, we did this by creating and computing three basis models. We then computed the electric field at the point of computation in each basis model. We combined these to produce an analytic expression for the electric field at the point of computation as the line voltages oscillated over time.

The second case study was a bit trickier than the first. Unlike in the first case study, in the second case study, the geometry changed over time. We were still able to use basis models to aid in the computation of the electric field. Instead of creating and computing a collection of basis models for the entire dynamic model, we did this for each frame. Using these, we were able to compute the charge distribution in each frame, and thus observe how the electric field behaved as the human walked.

The use of basis models provides significant savings in computation time when working with large dynamic models. Imagine we want to tackle the first case study without the use of basis models. Suppose we break the dynamic model into p distinct frames. We would have had to compute p right hand sides. This would have required $O(pn^2)$ operations, where n is the number of elements in the model. If p and n are in the ten thousands, then this amounts to teraflops of computation. In comparison, using the basis models approach, only $O(bn^2 + pb)$ operations are required to compute the electric field at a given location, where b is the number of basis models. This amounts to only gigaflops of computation and represents a computational savings of three orders of magnitude, reducing days of computation time into minutes, with no loss of accuracy.

REFERENCES

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